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Quaternionic representation of the Coxeter group $W(H_4)$ and the polyhedra

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Abstract

The vertices of the four-dimensional polytope {3, 3, 5} and its dual {5, 3, 3} admitting the symmetry of the non-crystallographic Coxeter group $W(H_4)$ of order 14 400 are represented in terms of quaternions with unit norm where the polytope {3, 3, 5} is represented by the elements of the binaryicosahedral group of quaternions of order 120. We projected the polytopes to three-dimensional Euclidean space where the quaternionic vertices are the orbits of the Coxeter group $W(H_3)$, icosahedral group with inversion, where $W(H_3) \times Z_2$ is one of the maximal subgroups of the Coxeter group $W(H_4)$. The orbits of the icosahedral group $W(H_3)$ in the polytope {3, 3, 5} are the conjugacy classes of the binary icosahedral group and represent a number of icosahedrons, dodecahedrons and one icosidodecahedron in three dimensions. The 15 orbits of the icosahedral group $W(H_3)$ in the polytope {5, 3, 3} represent the dodecahedrons, icosidodecahedrons, small rhombicosidodecahedrons and some convex solids possessing the icosahedral symmetry. One of the convex solids with 60 vertices is very similar to the truncated icosahedron (soccer ball) but with two different edge lengths which can be taken as a realistic model of the C_{60} molecule at extreme temperature and pressure.

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1. Introduction

The non-crystallographic Coxeter group $W(H_4)$ of order 14 400 generates interest [1–3] for its relevance to the quasicrystallography as well as to its unique relation to $W(E_8)$ [4, 5], the Weyl group of the exceptional Lie group E_8 which seems to be playing an important role in the superstring theory [6]. The Coxeter group $W(H_4)$ arises as the symmetry group of

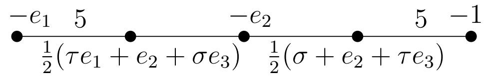

Figure 1. The extended Coxeter diagram of H_4 with scaled quaternionic simple roots.

the polytope {3, 3, 5} [7], the vertices of which can be represented by 120 quaternions of the binary icosahedral group [8, 9]. The dual of the polytope {3, 3, 5} is another polytope {5, 3, 3} with 600 vertices [7, 8] which can be represented by quaternions.

In this paper, we study the projections of these polytopes in three dimensions using one of the maximal subgroups of $W(H_4)$ [10]. The non-crystallographic Coxeter group $W(H_3) \times Z_2$ is one of those five maximal subgroups of $W(H_4)$ where $W(H_3)$, of order 120, acts in three-dimensional space and Z_2 is the generator of the root of the Lie algebra A_1 orthogonal to H_3. We organize the paper as follows. In section 2, we introduce the quaternionic root system of H_4 in which the roots of H_3 are represented by imaginary quaternions. We also discuss the 120 embeddings of $W(H_3)$ in the group $W(H_4)$. In section 3, we construct the quaternionic vertices of the polytope {5, 3, 3} and obtain the orbits of $W(H_3)$ as sets of quaternionic vertices of the polytopes {3, 3, 5} and {5, 3, 3}. We plot the polyhedra corresponding to the orbits of $W(H_3)$. Finally in section 4 we discuss our results regarding their relevance to other algebraic structures.

2. Construction of the Coxeter groups $W(H_4)$ and $W(H_3)$ in terms of quaternions

Let $q = q_0 + q_i e_i$, $(i = 1, 2, 3)$ be a real quaternion with its conjugate defined by $q = q_0 - q_i e_i$ where the quaternionic imaginary units satisfy the relations:

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k, \qquad (i, j, k = 1, 2, 3). \tag{1}$$

Here δ_{ij} and ϵ_{ijk} are the Kronecker and Levi-Civita symbols and summation over the repeated indices implicit. Quaternions generate the four-dimensional Euclidean space where the quaternionic scalar product is defined as

$$(p, q) = \tfrac{1}{2}(\bar{p}q + \bar{q}p). \tag{2}$$

The group of quaternions is isomorphic to $SU(2)$ which is a double cover of the proper rotation group $SO(3)$. The imaginary quaternionic units e_i can be related to the Pauli matrices σ_j by $e_j = -\mathrm{i}\sigma_j$ and the unit quaternion is represented by 2×2 unit matrix. The affine extension of the Coxeter diagram H_4 is depicted in figure 1. Here $\tau = \frac{1+\sqrt{5}}{2}$, $\sigma = \frac{1-\sqrt{5}}{2}$ satisfy the relations $\tau\sigma = -1$, $\tau + \sigma = 1$, $\tau^2 = \tau + 1$ and $\sigma^2 = \sigma + 1$.

Deleting the second root from right one obtains the quaternionic roots of the root system of $H_3 \oplus A_1$ where the imaginary roots represent the simple roots of H_3 and the quaternionic unit -1 stands for the simple root of A_1. Let us introduce the notations for the action of unit quaternions on an arbitrary quaternion q. We define the pair of quaternions representing the group elements of $O(4)$ [5] as

$$[a, b] : q \to q' = aqb \qquad [c, d]^* : q \to q'' = c\bar{q}d. \tag{3}$$

In this notation the generators of $W(H_3) \times Z_2$ would follow from figure 1 as

$$[-e_1, e_1]^*, \left[-\tfrac{1}{2}(\tau e_1 + e_2 + \sigma e_3), \tfrac{1}{2}(\tau e_1 + e_2 + \sigma e_3)\right]^*, [-e_2, e_2]^*, [-1, 1]^*. \tag{4}$$

The first three generators generate the group elements $[p, \bar{p}]$, $[p, \bar{p}]^* = [1, 1]^*[p, \bar{p}]$ with $p \in I$, where I represents the binary icosahedral group of order 120 and the elements are given in table 1. One can prove that the set of elements $[p, \bar{p}]$, $p \in I$ is isomorphic to the

Table 1. Conjugacy classes of the binary icosahedral group I represented by quaternions.

Conjugacy classes and order of elements	Elements of conjugacy classes (cyclic perm. of e_1, e_2, e_3 must be added if not included)
1	1
2	-1
10	$12_+ : \frac{1}{2}(\tau \pm e_1 \pm \sigma e_3)$
5	$12_- : \frac{1}{2}(-\tau \pm e_1 \pm \sigma e_3)$
10	$12'_+ : \frac{1}{2}(\sigma \pm e_1 \pm \tau e_2)$
5	$12'_- : \frac{1}{2}(-\sigma \pm e_1 \pm \tau e_2)$
6	$20_+ : \frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(1 \pm \tau e_1 \pm \sigma e_2)$
3	$20_- : \frac{1}{2}(-1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(-1 \pm \tau e_1 \pm \sigma e_2)$
4	$30 : \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm \sigma e_1 \pm \tau e_2 \pm e_3)$

icosahedral group A_5, the even permutations of five letters and the generator $[1, 1]^*$ commutes with $[p, \bar{p}]$ so that the group has the structure $W(H_3) \approx A_5 \times Z_2$. Since the generator $[-1, 1]^*$ commutes with the elements of the group $W(H_3) \approx A_5 \times Z_2$, then the group $W(H_3) \times Z_2$ has the structure

$$W(H_3) \times Z_2 \approx A_5 \times Z_2^2 = \{[p, \pm\overline{p}], [p, \pm\overline{p}]^*; \, p, \overline{p} \in I\} \tag{5}$$

with 240 elements. The Coxeter group $W(H_4)$ of order $120 \times 120 = 14400$ can be generated by reflections at hyperplanes perpendicular to its four simple roots which leads to the group structure given as follows:

$$W(H_4) = \{[p, q] \oplus [p, q]^*; \, p, q \in I\}. \tag{6}$$

It is clear that the group $W(H_4)$ is the symmetry group of the set of quaternions I which represent the vertices of the polytope $\{3, 3, 5\}$. A simple construction of the quaternions of I can be given as follows. Let us denote the elements of the binary tetrahedral subgroup of the group I by the quaternions [11],

$$T = \left\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\right\} \tag{7}$$

which also represent the vertices of the polytope $\{3, 4, 3\}$ [8] as well as the nonzero roots of $SO(8)$. Take any element $p^5 = \pm 1$ of I, say, $p = \frac{1}{2}(\tau + e_1 + \sigma e_3)$.

The set of quaternions $\sum_{j,k=1}^{5} \oplus p^j T \bar{p}^k$ constitutes the five copies of I [8]. Actually we can write the elements of I in either form $I = \sum_{j=1}^{5} p^j T = \sum_{k=1}^{5} T \bar{p}^k$. The five conjugate groups of T in I can be represented by $p^j T \bar{p}^j$, $p \in I$, $(j = 1, \ldots, 5)$. Now we discuss the 60 different embeddings of the group $W(H_3) \times Z_2$ defined in (5) in the group. It is clear that the group in (5) leaves the quaternionic unit vector ± 1 invariant. The conjugate groups of $W(H_3) \times Z_2$ in (5) can be obtained by performing the group conjugations:

$$[a, b][p, \pm\overline{p}][a, b]^{-1} = [ap\overline{a}, \pm\overline{b}\overline{p}b], \tag{8}$$

$$[a, b][p, \pm\overline{p}]^*[a, b]^{-1} = [apb, a\overline{p}b]^*. \tag{9}$$

If we define $q = a\bar{p}\bar{a}$ in (8) and $t = apb$ in (9) and let $c = ab$ then the group elements in (8) and (9) can be written as $[q, \pm\bar{c}\bar{q}c]$ and $[t, \pm c\bar{t}c]^*$, $(q, t, c \in I)$. Therefore without loss of generality we represent the conjugate groups by

$$W(H_3) \times Z_2 = \{[p, \pm\overline{c}\,\overline{p}c], [p, \pm c\overline{p}c]^*\} \tag{10}$$

where the element $\pm c$ is a fixed vector of I for each conjugate group but p is any element of I. It is straightforward to check that the group in (10) leaves the vector $\pm c$ invariant. Since we have 60 different choices of $\pm c$ from I the group representation of $W(H_3) \times Z_2$ in (10) is one of those 60 different embeddings of $W(H_3) \times Z_2$ in $W(H_4)$.

3. The orbits of $W(H_3)$ in the polytopes {3, 3, 5} and {5, 3, 3}

The vertices of the dual polytope {5, 3, 3} can be constructed from T', the dual of T that are the quaternions [11]:

$$T' = \left\{\tfrac{1}{\sqrt{2}}(\pm 1 \pm e_1), \tfrac{1}{\sqrt{2}}(\pm e_2 \pm e_3), \tfrac{1}{\sqrt{2}}(\pm 1 \pm e_2), \tfrac{1}{\sqrt{2}}(\pm e_3 \pm e_1), \tfrac{1}{\sqrt{2}}(\pm 1 \pm e_3), \tfrac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\right\}. \tag{11}$$

The union $O = T \oplus T'$ is the binary octahedral group, and $T'/\sqrt{2}$ not only represents the short roots of the exceptional Lie algebra F_4 but also constitutes the vertices of the dual polytope {3, 4, 3} [11]. Now the vertices of the polytope {5, 3, 3} can be constructed as follows:

$$\{5, 3, 3\} = \sum_{j,k=1}^{5} \oplus p^j T' \bar{p}^k. \tag{12}$$

Let $t' \in T'$ be an arbitrary element of (11). One can show that $T' = t'T = Tt'$. The set of points

$$\{3, 3, 5\}_j = \sum_{k=1}^{5} \oplus p^j T' \bar{p}^k = \sum_{j=1}^{5} \oplus p^j T t' T \bar{p}^k = p^j t' I \tag{13}$$

$$\{3, 3, 5\}_k = \sum_{j=1}^{5} \oplus p^j T' \bar{p}^k = \sum_{j=1}^{5} \oplus p^j T t' T \bar{p}^k = I t' \bar{p}^k, \tag{14}$$

each representing a copy of {3, 3, 5} in the polytope {5, 3, 3} so that the vertices of the dual polytope can be written as

$$\{5, 3, 3\} = \sum_{j=1}^{5} \oplus p^j t' I = \sum_{k=1}^{5} \oplus I t' \bar{p}^k. \tag{15}$$

One can prove that the set of vertices of the polytope {5, 3, 3} is invariant under the Coxeter group $W(H_4)$, the quaternionic representation of which is given in (6). To convince the reader that the quaternions in (12) are the vertices of the dual polytope {5, 3, 3} we give the following argument. The four quaternions of I

$$1, \tfrac{1}{2}(\sigma + e_1 - \tau e_2), \tfrac{1}{2}(\sigma + e_2 - \tau e_3), \tfrac{1}{2}(\sigma + e_3 - \tau e_1) \tag{16}$$

form the vertices of a tedrahedron of the polytope {3, 3, 5} which consists of 600 tetrahedra of this type. It is the reason that the polytope {3, 3, 5} is called 600-cell. Since the full symmetry of a tedrahedron is isomorphic to the symmetric group S_4 of order 24 and it can be embedded in the Coxeter group $W(H_4)$, $\frac{120\times120}{24} = 600$ ways the polytope {3, 3, 5} consists of 600 tedrahedrons. The vertices of the dual polytope {5, 3, 3} are obtained by extending the quaternions representing the centres of the tedrahedrons. When the average of the quaternions in (16) is extended to the unit sphere S^3, one obtains the quaternion $\frac{1}{2\sqrt{2}}(\sigma^2 - \tau e_1 - \tau e_2 - \tau e_3)$, which can be written as the product of two quaternions—one from T' in (11) and the other from I in table 1:

$$\tfrac{1}{2\sqrt{2}}(\sigma^2 - \tau e_1 - \tau e_2 - \tau e_3) = \tfrac{1}{\sqrt{2}}(1 + e_1)\tfrac{1}{2}(\sigma - e_1 - \tau e_2) \tag{17}$$

which is one of those quaternions in (12).

Before we determine the orbits of $W(H_3)$ in the set of quaternions representing the polytopes $\{3, 3, 5\}$ and $\{5, 3, 3\}$, we discuss the intersection of the sphere S^3 with the hyperplane which can be represented by the equation

$$c_0q_0 + c_1q_1 + c_2q_2 + c_3q_3 = \tfrac{1}{2}(\bar{c}q + \bar{q}c) = d, \tag{18}$$

where q is an arbitrary quaternion but c is a fixed quaternion. The equation of a hyperplane is obtained as the scalar product of q with c. Since we have chosen our quaternions representing the vertices of the polytopes as unit quaternions, they satisfy the equation

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1. \tag{19}$$

The intersection of this S^3 with the hyperplane in (18) is a quadric surface in general. In the special case where $c = 1$ the quadric surface is a sphere S^2 with radius $\sqrt{1 - d^2}$. As d varies we obtain a number of parallel hyperplanes intersecting with the sphere S^3 leading to different spheres S^2 with various radii. Since we talk about the discrete points on the sphere S^2 they will represent the polyhedra or convex solids in general in three dimensions.

3.1. The orbits of $W(H_3)$ in $\{3, 3, 5\}$

The orbits of $W(H_3)$ in $\{3, 3, 5\}$ are the conjugacy classes of the binary icosahedral group I shown in table 1. The elements ± 1 are single points not corresponding to any polyhedra. However the vertices in the conjugacy classes $12_{\pm}$ represent an icosahedron with the vertices

$$\tfrac{1}{2}(\pm e_1 \pm \sigma e_3),\ \tfrac{1}{2}(\pm e_2 \pm \sigma e_1),\ \tfrac{1}{2}(\pm e_3 \pm \sigma e_2) \tag{20}$$

with radius $\frac{\sqrt{2+\sigma}}{2}$. Actually the set of quaternions in (20) represents two icosahedra each lying in parallel hyperplanes with the values $d = \pm\frac{\tau}{2}$. However in three dimensions, they coincide. Similarly the set of imaginary quaternions in the conjugacy classes $12'_{\pm}$ with the vertices

$$\tfrac{1}{2}(\pm e_1 \pm \tau e_2),\ \tfrac{1}{2}(\pm e_2 \pm \tau e_3),\ \tfrac{1}{2}(\pm e_3 \pm \tau e_1) \tag{21}$$

represent two icosahedra with the radii $\frac{\sqrt{2+\tau}}{2}$ for $d = \pm\frac{\sigma}{2}$. The imaginary quaternions in the conjugacy classes $20_{\pm}$ with 20 vertices

$$\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3),\ \tfrac{1}{2}(\pm \tau e_1 \pm \sigma e_2),\ \tfrac{1}{2}(\pm \tau e_2 \pm \sigma e_3),\ \tfrac{1}{2}(\pm \tau e_3 \pm \sigma e_1) \tag{22}$$

represent two dodecahedra with the radius $\frac{\sqrt{3}}{2}$ where $d = \pm\frac{1}{2}$. The imaginary quaternions in the last conjugacy class 30 represent the 30 vertices of an icosidodecahedron with radius 1.

3.2. The orbits of $W(H_3)$ in $\{5, 3, 3\}$

There are 15 orbits of $W(H_3)$ in the polytope $\{5, 3, 3\}$. Seven of them are in pairs with varying $\pm d$ and one with $d = 0$. Before we discuss all the orbits one by one we note three special orbits with $d = 0$ and $d = \pm\frac{1}{\sqrt{2}}$. The subset of quaternions from (12)

$$\sum_{j=1}^{5} \oplus p^j T' \bar{p}^j \tag{23}$$

lies in three orbits as we will explain below. When we look at the quaternions of T' in (11) we see that they can be classified with respect to their scalar values $\left(\mathrm{Sc}\, q = \frac{1}{2}(q+\bar{q}) = q_0\right)$ as $\pm\frac{1}{\sqrt{2}}$ and zero. Since the sum in (23) does not change $\mathrm{Sc}\, q$ of the quaternions we can classify them as the quaternions with $\mathrm{Sc}\,q$ equal to zero, $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$. Twelve of the quaternions in (11) are with $\mathrm{Sc}\,q = 0$ and those sets with $\mathrm{Sc}\,q = \pm\frac{1}{\sqrt{2}}$ each constitutes a set of six quaternions. Therefore, the sets of quaternions in (23) are 60 with $\mathrm{Sc}\,q = 0$, 30 with $\mathrm{Sc}\,q = \pm\frac{1}{\sqrt{2}}$. The group $W(H_3)$ preserves these structures as it does not change $\mathrm{Sc}\,q$.

3.2.1. Orbits with $q_0 = 0$. The 60 vertices of the orbit of $W(H_3)$ with $\mathrm{Sc}\,q = 0$ are given by

$$\left\{\tfrac{1}{\sqrt{2}}(\pm e_1 \pm e_2), \tfrac{1}{2\sqrt{2}}(\pm e_1 \pm \tau^2 e_2 \pm \sigma^2 e_3), \tfrac{1}{2\sqrt{2}}(\pm(\tau-\sigma)e_1 \pm \sigma e_2 \pm \tau e_3)\right\} + \text{cyclic permutation of } e_1, e_2, e_3. \tag{24}$$

These vertices constitute a convex solid (not even classified among the semi-regular polyhedra) with two edge lengths, 12 pentagonal, 20 triangular and 30 rectangular faces. The widths of the rectangles are the edges of the pentagons and the lengths of the rectangles are the sides of the triangular faces. The length to the width ratio of the rectangle is $\frac{l}{w} = \tau^2$. It is depicted in figure 2(*a*).

3.2.2. Orbits with $q_0 = \pm\frac{1}{\sqrt{2}}$. The projection of the polytope on the hyperplanes with $q_0 = \pm\frac{1}{\sqrt{2}}$ results in the quaternionic vertices:

$$\left\{\pm\tfrac{1}{\sqrt{2}}e_1, \pm\tfrac{1}{\sqrt{2}}e_2, \pm\tfrac{1}{\sqrt{2}}e_3, \tfrac{1}{2\sqrt{2}}(\pm e_1 \pm \sigma e_2 \pm \tau e_3)\right\} + \text{cyclic permutation of } e_1, e_2, e_3. \tag{25}$$

These are the 30 vertices of the icosidodecahedron (one of those Archimedean solids) with 60 edges, 12 pentagonal and 20 triangular faces. All the edges are equal. Actually we have two polyhedra here: one in the hyperplane with $q_0 = \frac{1}{\sqrt{2}}$ and the other in $q_0 = -\frac{1}{\sqrt{2}}$. The circumscribed radius is $\frac{1}{\sqrt{2}}$ and it is shown in figure 2(*b*).

3.2.3. Orbits with $q_0 = \pm\frac{1}{2\sqrt{2}}$. The polytope $\{5, 3, 3\}$ has the following vertices on the hyperplanes $q_0 = \pm\frac{1}{2\sqrt{2}}$:

$$\left\{\tfrac{1}{2\sqrt{2}}\big(\pm\sigma^2 e_1 \pm \tau^2 e_2\big), \tfrac{1}{2\sqrt{2}}\big(\pm e_1 \pm e_2 \pm (\tau-\sigma)e_3\big), \tfrac{1}{2\sqrt{2}}(\pm\sigma e_1 \pm 2e_2 \pm \tau e_3)\right\} + \text{cyclic permutation of } e_1, e_2, e_3. \tag{26}$$

This is also a convex solid with 60 vertices and two different lengths of 90 edges also having 12 pentagonal faces and 20 non-regular hexagons. Its circumscribed radius is $\sqrt{\frac{7}{8}}$. Non-regular hexagons consist of two different lengths where the longer edges are shared with the pentagons and the shorter ones are shared among the non-regular hexagons. The ratio of the longer edge to the shorter one is the golden ratio $\tau \cong 1.618$. That would be the ideal model for C_{60} molecule if this ratio would be smaller. We know that the double C bond length is smaller than the single C bond length in the C_{60} molecule. The soccer ball model of C_{60} is not perfect as the edge lengths are equal in the truncated icosahedron. Since the bond lengths in the C_{60} molecule change with the pressure and the temperature, the molecule may change its shape between the soccer ball model and the model at hand. Its shape is depicted in figure 2(*c*).

3.2.4. Orbits with $q_0 = \pm\frac{\tau}{2\sqrt{2}}$. The projection of the polytope $\{5, 3, 3\}$ on the hyperplanes $q_0 = \pm\frac{\tau}{2\sqrt{2}}$ has a similar shape as above solid except the short and the long edges are interchanged. The ratio of the longer to the shorter edge is again the golden ratio τ. It has

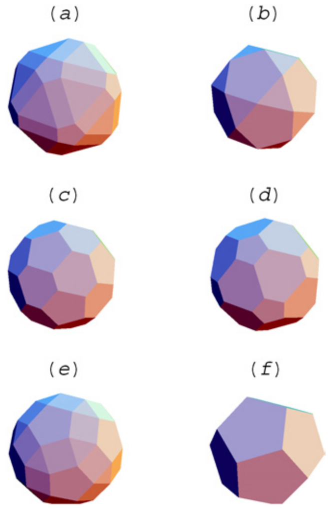

Figure 2. Ployhedra projected from the polytope {5, 3, 3} as orbits of $W(H_3)$.
(This figure is in colour only in the electronic version)

60 vertices, 90 edges, 12 pentagonal and 20 non-regular hexagonal faces. Its circumscribed radius is $\sqrt{\frac{7-\tau}{8}}$.The vertices are given by the quaternions

$$\left\{\frac{1}{2\sqrt{2}}(\pm\sigma e_1 \pm (\tau-\sigma)e_2), \frac{1}{2\sqrt{2}}(\pm e_1 \pm 2e_2 \pm \sigma e_3), \frac{1}{2\sqrt{2}}(\pm\sigma^2 e_1 \pm \tau e_2 \pm \tau e_3)\right\} + \text{cyclic permutation of } e_1, e_2, e_3. \tag{27}$$

Its shape is shown in figure 2(d).

3.2.5. Orbits with $q_0 = \pm\frac{\sigma}{2\sqrt{2}}$. This is a semi-regular polyhedron, which also has 60 vertices, 90 edges, 62 faces (12 pentagonal, 20 triangular, 30 square). It is known as the small rhombicosidodecahedron. Its circumscribed radius is $\sqrt{\frac{6+\tau}{8}}$. Its vertices are given by the quaternions:

$$\left\{\frac{1}{2\sqrt{2}}(\pm(\tau-\sigma)e_1 \pm \sigma e_2), \frac{1}{2\sqrt{2}}(\pm 2e_1 \pm e_2 \pm \tau e_3), \frac{1}{2\sqrt{2}}(\pm\sigma e_1 \pm \sigma e_2 \pm \tau^2 e_3)\right\} + \text{cyclic permutation of } e_1, e_2, e_3. \tag{28}$$

Its shape is shown in the figure 2(e).

3.2.6. Orbits with $q_0 = \pm\frac{\tau-\sigma}{2\sqrt{2}}$. This is a regular dodecahedron with 20 vertices, 30 edges and 12 pentagonal faces. Its circumscribed radius is $\sqrt{\frac{3}{8}}$. Its vertices are given by the quaternions:

$$\left\{\frac{1}{2\sqrt{2}}(\pm\tau e_1 \pm \sigma e_2), \frac{1}{2\sqrt{2}}(\pm e_1 \pm e_2 \pm e_3)\right\} + \text{cyclic permutation of } e_1, e_2, e_3. \tag{29}$$

Its shape is shown in figure 2(f).

3.2.7. Orbits with $q_0 = \pm\frac{\tau^2}{2\sqrt{2}}$. Here we have two more dodecahedra with the vertices:

$$\left\{\frac{1}{2\sqrt{2}}(\pm e_1 \pm \sigma^2 e_2), \frac{\sigma}{2\sqrt{2}}(\pm e_1 \pm e_2 \pm e_3)\right\} + \text{cyclic permutation of } e_1, e_2, e_3. \tag{30}$$

It has the same shape as in figure 2(f). Its circumscribed radius is $\frac{1}{\tau}\sqrt{\frac{3}{8}}$. Its vertices can be obtained from those in (27) by multiplying them by $\pm\sigma$. It is the reduced version of the dodecahedron discussed above.

3.2.8. Orbits with $q_0 = \pm\frac{\sigma^2}{2\sqrt{2}}$. The dodecahedra here is the rescaled versions of the dodecahedra with the vertices given in (27). The vertices here are the $\pm\tau$ times those in (27). The circumscribed radius is $\tau\sqrt{\frac{3}{8}}$.

$$\left\{\frac{1}{2\sqrt{2}}(\pm\tau^2 e_1 \pm e_2), \frac{\tau}{2\sqrt{2}}(\pm e_1 \pm e_2 \pm e_3)\right\} + \text{cyclic permutation of } e_1, e_2, e_3. \tag{31}$$

4. Conclusion

We have used the subgroup $W(H_3)$ of the Coxeter group $W(H_4)$ to project the four-dimensional polytopes {3, 3, 5} and {5, 3, 3} in three dimensions. The vertices of the convex solids in three dimensions are the orbits of the Coxeter group $W(H_3)$. One of the convex solid is very similar to the truncated icosahedron but with two different edge lengths. Since the C_{60} molecule displays different bond lengths at different pressure and temperature, we anticipate that the convex solids obtained in the hyperplane $q_0 = \pm\frac{1}{2\sqrt{2}}$ could be used as a model of C_{60} at some extreme temperature and pressure. This solid with 60 vertices and the truncated icosahedron may correspond to two extreme models of the C_{60} molecule where the soccer ball model corresponds to equal length bonds and that we discussed gives the ratio of the single bond to double bond lengths as the golden ratio τ.

We also know that the Coxeter group $W(H_4)$ is one of the maximal subgroups of the Weyl group $W(E_8)$. The quasi-lattice structure of H_4 can be embedded in the E_8-lattice. The projection of the E_8-lattice to the four-dimensional Euclidean space via $W(H_4)$ and then to three-dimensional Euclidean space by the Coxeter group $W(H_3)$ may yield a rich structure of convex solids some of which, as we have already seen, may correspond to regular and semi-regular polyhedra.

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